Using Index ETFs for Multi-Asset-Class Investing: *Shifting the Efficient Frontier Up*

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This article seeks to extend classic investing, which is often limited to only domestic equities or to a mix of equities and bonds, to a wider array of lower-correlation non-equity assets. The ready availability of highly liquid index exchange-traded funds (ETFs) on assets such as domestic equities, international equities, treasury/sovereign bonds, real estate, gold bullion, and foreign currencies has the potential to extend our availability set and the resulting risk–return (σ–μ) efficient frontier beyond what is possible with only equity-only portfolios. This article shows that. This could take us a step closer toward meeting the all-inclusive, but elusive, "true market portfolio" (Roll [1977]) and thinking "out of the *equity* box" that we seem to be perpetually trapped in.

Investors stand to gain from the additional diversification made feasible by extending into a multi-asset-class (MAC) portfolio-based covariance matrix. The shrinkage of the asset covariance structure (Choueifaty, Froidure, and Reynier [2013]) and an overall reduction in the cross-correlations of the constituent assets (Willenbrock [2011]) have the potential to produce efficient frontiers that would not be possible with pure-equity-based portfolios. Eventually, that would be an efficiency gain for the investors.

The article uses the Markowitz [1956] mean–variance optimization (MVO) process to show that the efficient frontier of a MAC portfolio dominates not only the capitalization-weighted Russell 1000 Index but an all-equity mean–variance optimization (MVO) efficient frontier as well. The risk-adjusted Sharpe ratios (Sharpe [1987]) and the beta of the MAC portfolio are developed and reported as well. Furthermore, in addition to the MVO portfolios, a 1/N equal-weighted portfolio is created from the constituents of the MAC portfolio and plotted in the σ–μ space. The alternate portfolios are then tested for relative portfolio efficiency by the application of the exact Gibbons, Ross, and Shanken [1989] GRS W-test.

LITERATURE AND DATA

Besides risk–return efficiency, a tradable implementation of the diversified MAC portfolio approach was also a consideration in the design of this study. To enable that, a set of highly liquid ETFs were selected (Agrrawal and Clark [2009]) that represented six major asset classes. Roll [2013] wrote that "across asset classes, ETF heterogeneity might be acceptable… though it is not that impressive within each class" and discussed the equity, bond, commodities, and currency asset classes. The Blake, Lehmann, and Timmermann [1999] strategic allocation study on U.K. pension funds included real estate as an additional asset class. Black and Litterman [1992] showed how quantitative asset allocation models could significantly improve global capital asset pricing model (CAPM) equilibrium given the "straightforward mathematics of this optimization." Booth and Fama [1992] showed that diversification through asset allocation increases compound returns by dampening return volatility and that maintaining constant weights is important to capture the increased returns (in effect, cautioning against market timing or tactical asset allocation). Campbell, Chan, and Viceira [2003] discussed how predictability of asset returns could affect portfolio choices of long-lived investors who value wealth for the consumption it supports.

The ETFs selected for this study are relatively recent but have continuous daily pricing history available since December 31, 2004 (obtained from the CRSP/WRDS Wharton dataset). This period includes one full bear market (2007–2009) and two bull markets (2004–2007 and 2009–May 2013, at the time of this writing), thus insulating the study from explicit market-phase bias. These six highly liquid ETFs (Exhibit 1) have a wide coverage of the broader asset classes and constitute a globally diversified multi-asset portfolio, representing U.S. equities, MSCI-EAFE and MSCI-Emerging markets, the U.S. real estate sector, U.S. Treasuries (long term), gold bullion, and foreign currencies (indirectly through the un-hedged equity exposure to Europe, Asia, Far East—or EAFE—and emerging equity markets); a similar covariance matrix reduction approach was deployed by Lee [2011] and can be thought of as diversification within diversification (Exhibit 2 has the correlation matrixes). In addition, ETFs for the Russell 1000 and the risk-free proxy (ETF ticker BIL, comprising U.S. Treasury bills) are used for constructing the minimum-variance efficient frontier that is independent of expected returns (unlike a mean–variance frontier—Clarke, de Silva, and Thorley [2006]) and calculating performance ratios and market betas. A nonsingular invertible variance–covariance matrix Σ^{-1} for mean–variance optimization purposes is also ensured with this setup. With $N = 6$ assets, $N(N-1)/2$ covariances, no perfect asset correlations, $\rho_{ij} \neq +1$, and $N < T$; where N is the number of securities and T the number of returns per security, the optimizations are computationally very feasible and not onerous (despite having over 1,000 underlying securities). Such a matrix is also $N \times N$ symmetric, positive definite, with variances σ_I^2 on the main diagonal and covariances $\sigma_{IJ} = \rho_{IJ}\sigma_I\sigma_J$ on the off diagonals. In Exhibit 1 we present some of portfolio metrics associated with the constituents of the MAC diversified portfolio.

GENERALIZED VARIANCE OF A MAC PORTFOLIO VERSUS AN ALL-EQUITY PORTFOLIO

The determinant of the variance–covariance matrix $|\Sigma|$, or the generalized variance (GV) measures the spread across the variables (assets). It can be thought of as a scalar construct of overall dependencies and used

EXHIBIT 1
Assets of the MAC Portfolio

Asset Name	Ticker	Market Cap	Average Trading Volume	Beta (3 year, weekly)	Beta (full period, weekly)	Annual Volatility	Annual Total Returns	Correlation with Russell 1000
EAFE Index	EFA	$42B	18M	1.123	1.038	22.57%	4.4%	0.90
Emerging Markets	EEM	$44B	49M	1.114	1.253	29.24%	9.6%	0.85
S&P 500 Index	SPY	$140B	133M	1.000	1.000	19.66%	5.90%	1.00
U.S. Long-Term Treasuries	TLT	$3.5B	5.4M	**–0.67**	–0.32	13.76%	7.4%	–0.46
Real Estate	IYR	$5.7B	8.1M	0.98	1.21	30.12%	6.0%	0.81
Gold Bullion	GLD	$45B	10M	**0.21**	0.04	19.92%	14.0%	0.05
Russell 1000	IWB	$7.8B	0.71M	1.01	0.99	19.64%	5.93%	1.00
U.S. T-Bills	BIL	$1.16B	0.42M	–0.01	–0.01	0.8%	0.7%	–0.29

Notes: IWB and BIL are used as market and risk-free proxy benchmarks; they are not part of our MAC portfolio constituents. Notice that their liquidity, although good, is not of the same order as the first six MAC assets. Beta is versus the Russell 1000 (Agrrawal and Waggle [2010]). The returns are over the period from 12/2004 to 5/2013, the earliest that pricing existed for each of the ETFs. Full-period betas are included to give an idea of the beta coefficient drift.

EXHIBIT 2
Correlation Matrixes (12/2004–5/2013)

MAC Portfolio Correlations						
ΣMAC	**EFA**	**EEM**	**SPY**	**TLT**	**IYR**	**GLD**
EFA	1					
EEM	0.893	1				
SPY	0.904	0.842	1			
TLT	**−0.439**	**−0.411**	**−0.451**	**1**		
IYR	0.697	0.691	0.789	**−0.301**	1	
GLD	0.204	0.244	0.041	**−0.030**	0.056	1

**All correlations significant at $p < 0.01$

All Equity Portfolio Correlations						
Σequity	**IWB**	**EFA**	**IWM**	**IWO**	**IWN**	**DFBMX**
Russell 1000	1					
MSCI—EAFE	0.902	1				
Russell 2000	0.942	0.845	1			
Russell 2000 Growth	0.935	0.849	0.989	1		
Russell 2000 Value	0.930	0.827	0.989	0.963	1	
High Book-Market	0.970	0.879	0.914	0.899	0.912	1

**All correlations significant at $p < 0.01$

Notes: All of these ETFs had continuous daily pricing data since 12/2004; they represent the indexes of broad asset classes and are chosen for their representativeness, tradability, and ease of use in large-portfolio formation due to being highly liquid. The cross-correlations of the MAC portfolio are in general lower relative to the equity-only portfolio, which itself is diversified across various size, style, and country demarcations. This leads to efficiency gains in the risk–return space.

as a measure of the overall spread of the distribution (Kim and Bera [2012]). It is zero if any two assets are perfectly correlated and increases in value as off-diagonal asset covariances (correlations) drop. So a higher GV is desirable for the purposes of diversification.

As a simple example, suppose we have a matrix $\sum = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$, then it follows that:

$$\det\Sigma \text{ or } |\Sigma| = 1.1 - x.x = 1 - x^2$$

- If $x < 1$ then increasing the off-diagonal entries decreases the determinant.
- For $x = 1$, which is the case of perfect correlations, the generalized variance $|\Sigma| = 0$ and the Σ is singular and hence non-invertible, from which asset weights cannot be calculated.
- For $x = 0$, which is the case of no correlation, the generalized variance $|\Sigma| = 1$, the highest.
- For $x < 1$ when the off-diagonal entries decrease in value then the determinant value rises, which can be interpreted as higher generalized variance for the assets under consideration, as asset correlations drop, because Σ_{ij} is bounded ±1. This is the interesting case for the MAC diversified portfolio, because its cross-correlations are typically lower than those for an equity-only portfolio (see Exhibit 2).

The determinant of a 2×2 var–cov matrix Σ can also be given a geometrical interpretation—it is the area of the parallelogram defined by each of the two vectors that comprise Σ, the var–cov matrix. There are two special cases: one if the vectors are perpendicular, resulting in the value of the determinant being equal to the area of a rectangle, also called orthogonal vectors or uncorrelated factors; the second, if both the vectors are perfectly overlapping or linearly dependent on each other, in which case the value of the determinant is zero, and the parallelogram collapses to a point with no area. In other words, for two return vectors x_1 and x_2, the spanned area is $\det \sum = \|x_1\| \|x_2\| \sin(\alpha)$, which would be zero if $\alpha = 0$ and maximum if the angle $\alpha = 90°$, because $\sin(90°) = 1$.

As an example, we compare the GV given by the determinant $|\Sigma|$ of a six-asset MAC portfolio with that of a highly diversified "all equity" portfolio (Booth and Fama [1992]). This equity-only portfolio, separate from the MAC portfolio, comprises ETFs that represent the Russell 1000, MSCI EAFE, Russell 2000 Small Cap, Russell 2000 Small Cap Growth, Russell 2000 Small Cap Value, and the DFA High Book to Market fund.[1] Such a portfolio would typically be considered diversified across major equity markets, as well as size, style, and book-to-market indexes, besides having a very large intrinsic ***N***, because of the underlying securities that compose each index ETF.

Note that a correlation matrix is simply a normalized covariance matrix $\sigma_{IJ}/\sigma_I\sigma_J = \rho_{IJ}$ with standard deviations of the assets in the denominator and can be used without any loss of generality (Levy and Roll [2013]). Applying the determinant to each of the two matrixes results in the following:

$$|\Sigma_{MAC}| = 8.23\text{E-}03 > |\Sigma_{\text{All Equity}}| = 2.83\text{E-}07 \quad (1)$$

Equation (1) indicates that the extent of overlap in the $\Sigma_{\text{All Equity}}$ portfolio is much higher, resulting in a lower generalized variance (which for asset diversification is not a desirable feature). The implication of such a large difference in the generalized variance of the correlation structures of two very different set of assets is further explored with MVO efficient frontiers and an application of the GRS-W test (Gibbons, Ross, and Shanken [1989]) against the backdrop of the findings in the Levy and Roll [2010] market portfolio reverse-engineering paper.

MVO EFFICIENT FRONTIERS OF THE MAC DIVERSIFIED AND THE ALL-EQUITY PORTFOLIO

In this section, we use actual return data over the period December 31, 2004, to May 22, 2013 (the period over which continuous daily data for all 11 ETFs existed; EFA is common between the two portfolios), to generate two efficient frontiers by utilizing the Markowitz mean–variance optimization process.[2] The mean–variance efficient portfolio selection problem is one where the investor seeks to minimize the portfolio variance subject to the budget and target return constraint. A short-selling non-negativity constraint is optional, depending on the model. In this exercise the non-negativity constraint is applied.[3] Simply stated, the MVO problem is to

Minimize $\sigma^2(x) = x^T\Sigma x$
subject to $x^T e = 1$
where

$$e^T = [1,1,\ldots 1]$$
$$x^T\mu = \mu_p$$
$$x \geq o \ \text{(optional)} \tag{2}$$

where

- μ and x are N-vectors composed of asset rates of return and portfolio weights, respectively;
- Σ is an $N \times N$ positive-definite non-singular covariance matrix (the positive definiteness of Σ ensures that the value of the quadratic norm $\sigma^2(x)$ will be positive for all $x > 0$, essentially ensuring a positive variance (Greene [1993]);
- e is a unit vector composed of ones,
- and μ_p is a scalar equal to the targeted portfolio return or the maximum return possible at each level of risk.

In the MAC diversified portfolio, we have representation of generally accepted broad asset classes—U.S. equities (SPY), international equities (EFA and EEM), U.S. Treasuries (TLT), gold (GLD), (real estate IYR), and currency (indirectly through EAFE and emerging markets unhedged equity positions). An Ibbotson-NAREIT [2006] report showed the improvement in the risk–return tradeoff by including real estate investment trusts (REITs) in an optimal asset-allocation process.

We also construct a separate "All Equity" portfolio (also referred to here as an equity-only portfolio), diversified across global equity markets as well as size, style, and book-to-market indexes, to compare its risk-return features with the MAC portfolio. The equity-only portfolio comprises index ETFs that represent the Russell 1000, MSCI EAFE, Russell 2000 Small Cap, Russell 2000 Small Cap Growth, Russell 2000 Small Cap Value, and the DFA High Book to Market fund.[4]

The results of the optimization on these assets using returns over the December 2004–May 2013 period are shown in Exhibit 3. An efficient frontier was created using each set of returns (see Exhibit 4 for the frontiers). The optimizations were run independently to determine two separate efficient frontiers, one for the MAC diversified portfolio and the other for the All Equity portfolio. The minimum-variance (min-var) portfolio would be the left-most point on the frontiers in Exhibit 4. It can be seen that the MAC diversified portfolio dominates the All Equity portfolio. The GRS-W test (Exhibit 3 and Appendix A) also confirms that the minimum-variance points on each of the frontiers are significantly apart and the H_0 of the comparative efficiency of the All Equity minimum-variance point relative to the minimum-variance portfolio on the MAC frontier can be rejected. The visual separation (Exhibit 4) is confirmed by applying the Gibbons, Ross, and Shanken [1989] GRS-W statistic, given as follows:

$$W = \left[\frac{\sqrt{1+\hat{\theta}_*^2}}{\sqrt{1+\hat{\theta}_p^2}}\right]^2 - 1 \equiv \psi^2 - 1 \tag{3}$$

With a P-value of 1.33 E-30, the null hypothesis of portfolio efficiency (for the equity-only portfolio) is easily rejected (see Appendix A for details). The relative portfolio efficiency of the Russell 1000 location is similarly rejected (GRS P-value = 1.72 E-31). As can be seen

Exhibit 3

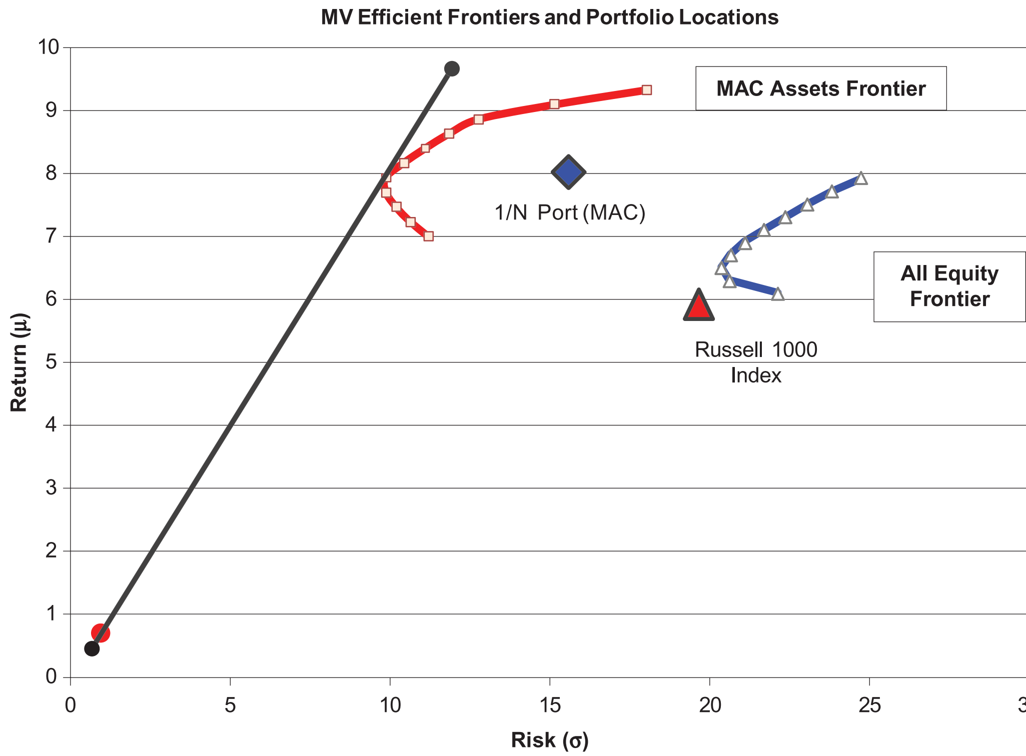
MAC Minimum Variance, All Equity Minimum Variance, Russell 1000, and 1/N Portfolios

	Portfolio Std Dev (σ)	Portfolio Return (μ)	Portfolio Beta (β)	Sharpe Ratio	GRS-W Test	GRS-W Test P-value
MAC Portfolio (Min-Var)	9.83	7.93	0.25	**0.736**	Reference tangency point	—
Equity Only Portfolio (Min-Var)	20.38	6.51	1.02	**0.285**	Significantly distinct	1.33 E-30
Russell 1000 Portfolio	19.65	5.93	1.00	**0.266**	Significantly distinct	1.72 E-31
1/N Portfolio (MAC constituents)	15.56	8.03	0.68	**0.471**	Significantly distinct	1.86 E-19

Notes: This exhibit gives the portfolio beta, return and volatility associated with each of the three quantitative portfolios, and the Russell 1000. Three of these are MAC diversified portfolios with varying asset allocations. The Sharpe ratio and the GRS-W test statistic for measuring portfolio efficiency are also listed. The GRS-W test indicates that all three portfolios are significantly away from the efficiency frontier, implying that they are not close to the tangency point. Appendix A gives additional parameters necessary for the test.

Exhibit 4

MVO Frontiers and Locations of MAC and Other Portfolios



Notes: The actual efficient frontiers are traced out using the Markowitz optimization process. The three large solid points show the locations of portfolios that are not part of the two MVO efficient frontiers, but which have been included for comparison purposes. The frontier comprising min-var optimized weights on the MAC assets is the one to the upper left. It dominates the frontier which comprises equity-only assets. This study shows that even a 1/N portfolio can dominate a pure equity frontier if its constituents are non-equity in composition; no ex ante return estimates are needed. Consistent with literature, the cap-weighted Russell 1000 index, although close to the equity-only frontier appears suboptimal relative to most other points on the exhibit. The circular dot near the origin is the proxy for the U.S./risk-free rate (BIL), with μ–σ of (0.7%, 0.8%).

from Exhibit 3, the Sharpe ratio for the All Equity portfolio declines to 0.285 relative to 0.736 for the min-var MAC portfolio.[5] This results from a 1.42% a year return spread in favor of the MAC diversified portfolio; additionally the portfolio risk is more than 50% lower than that of the All Equity portfolio. This efficiency gain is not the result any tactical allocation strategy or higher loadings on some new risk factor, rather it is the result of extending the capital allocation beyond the traditional single asset class of equities. Although it is good to see a positive return spread, the additional item of interest is the steep decline in portfolio risk (9.83% versus 20.38%). In Exhibit 3, two additional portfolio location points are included that are not on either frontier, these are the Russell 1000 and the 1/*N* portfolio. These are also plotted in Exhibit 4. The location of the cap-weighted Russell 1000 index ETF is marked as a triangle, it has the lowest return to risk ratio (μ/σ) and is to the bottom right on the risk–return plane (2% a year underperformance relative to the min-var MAC portfolio point, shown in Exhibit 3). As is to be expected, a naïve 1/*N* strategy of equal weighting the constituents of the MAC portfolio generates a risk–return location that plots within the MAC efficient frontier (diamond shaped point, Exhibit 4), however it may be noted that it is still above the All Equity efficient frontier and with a higher Sharpe ratio of 0.471 (Exhibit 3). An application of the GRS-W test to these points determines them to be significantly different from the reference MAC tangency point ($W = \psi^2 - 1 \rightarrow 0$ implies efficiency; see Exhibit 3, Exhibit 4, and Appendix A for additional details on the test).

As can be seen, the μ–σ parameters of other MVO efficient portfolios are higher than what can be attained for the minimum-variance cases; however, it is not the intention of this study to identify a particular combination of optimal asset weights, rather to demonstrate the significant shifting up of the entire frontier upon the inclusion of lower correlation asset classes, such as bonds, currencies, real estate, and gold relative to equities, both domestic and international. Although MVO methods on equity-only portfolios most definitely improve the risk–return tradeoff relative to the cap-weighted Russell 1000 equity index, the inclusion of low-correlation, non-equity assets shifts the frontier further up and left, leading to efficiency gains in the σ–μ space.

Additionally, the impact of omitting non-equity assets leads to reduced capital allocation to asset classes that derive a significant portion of their total return from less-volatile income assets, such as bonds or real-estate (Agrrawal and Borgman [2010]), which affects portfolio volatility (see standard deviations in Exhibit 3). Thus the use of equity-only portfolios leads to a frontier that has a lower return–risk ratio throughout and appears inferior to the set of corresponding points on an efficient frontier based on MAC assets (Exhibit 4). In a certain way, MAC diversification extends the traditional equity-only diversification to lower correlation asset-classes. A portfolio of MAC assets has an intrinsically lower variance–covariance structure, besides having a reduced size covariance matrix (Lee [2011]), both of which are desirable attributes. Erb and Harvey [2006] showed that the diversification return of a portfolio increases as the average correlation $\bar{\rho}$, of the $(K)(K-1)$ pair of assets in the portfolio declines and the average variance σ^2 rises. This is given as follows:

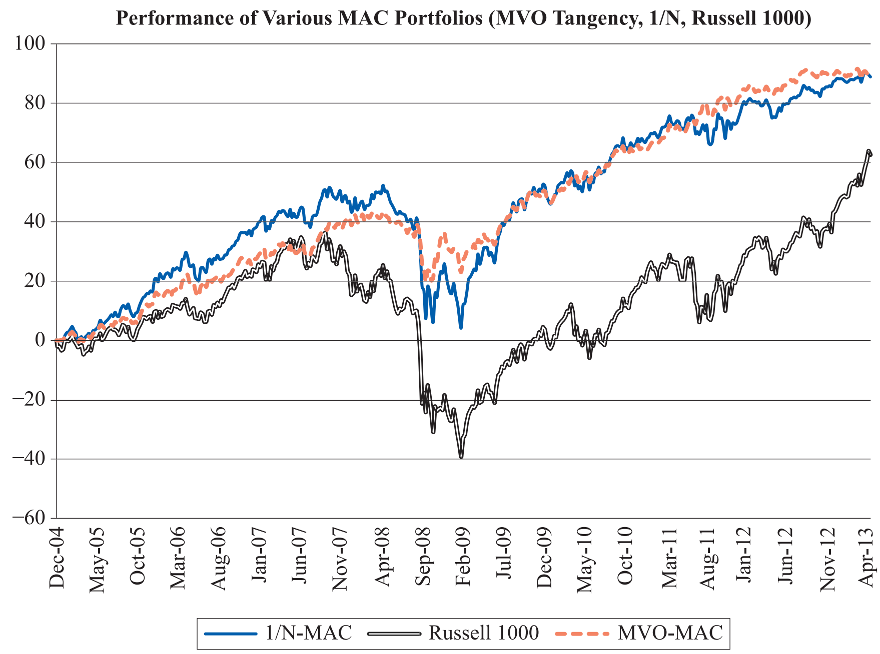
$$\text{Portfolio diversification return} = \frac{1}{2}(1-\bar{\rho})\left(1-\frac{1}{K}\right)\overline{\sigma^2} \tag{4}$$

The implication of diminished correlations is further explored in Willenbrock [2011], who showed that diversification returns persist even if $\bar{\rho}$ equals 1 and is attributed to rebalancing activity and mean-reversion of assets. Choueifaty and Coignard [2008] also added that the diversification ratio (DR) "increases when the average correlation and/or the CR (concentration ratio) decreases." These ratios are defined as: $DR(w) = [\bar{\rho}(w)(1 - CR(w)) + CR(w)]^{-1/2}$ for a vector of portfolio weights w and $CR(w) = \frac{\sum_i (W_i\sigma_i)^2}{\left(\sum_i W_i\sigma_i\right)^2}$; where ρ_{ij} and σ_i are the individual correlations and volatilities, respectively. As has been shown earlier (in Equation (1)), the generalized variance (GV) of the MAC portfolio is higher than the GV of a equity-only portfolio; this results from lower cross-correlations among the MAC assets (Exhibit 2 and the associated section).

PERFORMANCE OF TWO MAC PORTFOLIOS VERSUS THE RUSSELL 1000

The empirical performance of two MAC-based portfolios is presented in Exhibit 5 along with the performance of the Russell 1000 for the December 2004–May 2013 period.[6] In concurrence with the effi-

EXHIBIT 5
Return Performance Chart



Notes: The MAC portfolios' empirical performance is shown; all portfolios outperform the market, which in this case is the Russell 1000, and this is consistent with the documented inefficiency of the cap-weighted indexes (Choueifaty, Froidure, and Reynier [2013]). The volatilities of the Russell 1000 equity index and the 1/N portfolio are higher than that of the MVO-MAC portfolio. Nonetheless, the volatility of the 1/N MAC portfolio is still much lower than the market index.

cient frontier and portfolio location points as seen in Exhibit 4, the MAC 1/*N* equal asset weights portfolio has the highest return (88.91%), followed by the MAC tangency portfolio (88.31%) and then the Russell 1000 index (62.63%). The benchmark Russell 1000 was also the most volatile of the three and underperformed the other two portfolios in the risk–return space. Both MAC portfolios outperformed the equity market, which in this case was the Russell 1000, "this is consistent with the documented inefficiency of the cap-weighted indices," (Choueifaty, Froidure, Reynier [2013]). The MAC diversified portfolio performed well on both dimensions despite the equity market performing well over the period, so it cannot be the case that the MAC portfolio's performance was a function of a flat or negatively sloping equity return period[7] (which was not even the case).

The negative correlation of bonds and gold prevented the MAC portfolios from the severe erosion in value that the Russell 1000 suffered from during the 2008 financial crisis. The beta of the MAC tangency portfolio was a low 0.25 while the MAC 1/*N* portfolio had a higher beta of 0.68 (Exhibit 3). This translated into lower volatility, for both the portfolios, something that we see in Exhibits 4 and 5 for the corresponding assets.

FURTHER RESEARCH

The estimation errors associated with the ex ante formation of expected return vectors has often led to the well-known problem of error maximization in a mean–variance optimization setup (Michaud [1989]). MVO portfolios by design have lower volatility, however, they suffer from the risk of excessive asset concentration (corner solutions). Maximum risk diversification with a given variance–covariance structure Σ instead relies on such techniques as the risk-parity approach, which focuses on risk-based asset allocation schemes that equalize risk across the asset classes by varying individual asset allocation weights (Qian [2006]). Mathematically, for the bivariate $n = 2$ asset case,[8] where ρ_{12} is the correlation, σ_1, σ_2, and σ_p are the standard deviations of the assets and the portfolio, and non-negative weights such that $w_1 + w_2 = 1$, the problem reduces to equating the percentage contributions to total risk, $PCTR_i = PCTR_j$ $\forall i \neq j$:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + +2w_1w_2\rho_{12}\sigma_1\sigma_2$$

$$PCTR_1 = \left[w_1 \frac{\partial\sigma_p}{\partial w_1} \Big/ \sigma_p \right] = \frac{w_1^2\sigma_1^2 + w_1w_2\rho_{12}\sigma_1\sigma_2}{\sigma_p^2} \tag{5}$$

$$PCTR_2 = \left[w_2 \frac{\partial\sigma_p}{\partial w_2} \Big/ \sigma_p \right] = \frac{w_2^2\sigma_2^2 + w_1w_2\rho_{12}\sigma_1\sigma_2}{\sigma_p^2}$$

The interested reader is also directed to Benartzi and Thaler [2001], Choueifaty and Coignard [2008], Lee [2011], Chow, Hsu, Kalesnik, and Little [2011], and others listed in the References for significant work in the relatively new area of risk-based asset allocation. It would be interesting to see, and a topic for further research, how a MAC-based risk-parity portfolio compares to $1/N$ and the other portfolio variations discussed in this article. Regime dependency has to be looked into as well, although the results of this study cover the entire period for which the MAC assets were traded, so in that sense, it is a factual reporting of what actually existed.

CONCLUSION

How much of the MAC portfolio's performance is regime dependent still has to be seen and researched. The annualized return on bonds was 7.4% over this period compared with 5.9% for the S&P 500 (Exhibit 1). The only two asset classes that outperformed bonds over the period were gold (14%) and emerging markets (9.6%). However, relative performances seem to have changed since the beginning of 2013. U.S. equities have outperformed the two low-correlation assets—bonds and gold—this has happened along with a major weakening of the Japanese yen (which would negatively impact the unhedged EFA and thus the MAC portfolio's return).[9] Since no return expectations are built into a risk-based asset allocation scheme, however, such information can be considered external and irrelevant to portfolio architecture (but may not be to the market agents). It is in fact to dissociate the portfolio construction process from embedded return forecasts or volatility switching mechanisms that a MAC diversified approach is undertaken, which at its very core is a form of risk-based asset allocation. The naïve $1/N$ approach (DeMiguel, Garlappi, and Uppal [2009]) may be the preferred option for those investors who are averse to concentrated weights in low-correlation assets (such as bonds or commodities); notice that the asset weights of the MVO MAC portfolio are less dispersed compared to the $1/N$ portfolio (see Appendix B), or the seemingly esoteric nature of Markowitz optimization routines and related concerns regarding error maximization (Michaud [1989]).

In summary, apart from the usefulness of the $1/N$ portfolio in a multiple-asset allocation scheme (beyond stocks and bonds), the article also showed that a MAC diversified portfolio and the associated MVO frontiers can be easily and robustly constructed with off-the-shelf ETFs that produce a variance–covariance matrix with lower overlaps and no redundancies. We also showed that there are significant efficiency gains to be harvested if the investor chooses to step outside of the equity-only schema. This is established by the application of the Gibbons–Ross–Shanken [1989] GRS-W statistic. The result is free of any embedded vector of return forecasts or dynamic volatility-switching mechanisms. We also utilized a scalar construct of overall dependencies called generalized variance, which is a measure of the overall linear independence of the return vectors underlying a variance–covariance matrix. It condenses the $N(N-1)/2$ off-diagonal elements of the Σ matrix into a single scalar element measuring the degree of factor independence; we have not seen its utilization in mainstream finance literature. Finally, the article provided evidence to show that a MAC (multi-asset-class) diversified portfolio performed well in Markowitz mean–variance space and under varying market conditions, including the very adverse 2008 market crash and the bull markets preceding and subsequent to the 2008 meltdown.

APPENDIX A

THE GRS STATISTIC: GEOMETRICAL TEST FOR PORTFOLIO EFFICIENCY

Gibbons, Ross, and Shanken [1989] devised an exact form statistic to test for the MV efficiency of a given portfolio based on its geometric properties. The test is widely used in studies addressing the issue of portfolio efficiency and CAPM deviations (Zhou [1993] and Fama and French [2012], among others).

Exhibit A1

GRS-W Estimation Parameters for Test Portfolios

	Tangency MAC Portfolio (*)	Russell 1000 Index	1/N Portfolio	MVO Equity-only Portfolio
μ	7.93	5.93	8.03	6.51
Rf	0.70	0.70	0.70	0.70
σ	9.83	19.65	15.56	20.38
θ = (μ − Rf)/σ	**0.736**	**0.266**	**0.471**	**0.285**
GRS-W		0.439	0.261	0.425
N		6	6	6
T		437	437	437
Xf		31.61	18.80	30.61
P-value		**1.72E-31**	**1.86E-19**	**1.33E-30**
Significant P-value?		Yes	Yes	Yes
Close to Tangency?		No	No	No

The GRS statistic measures the distance in mean standard deviation space between a test portfolio (market index) and a tangency portfolio (on the efficient frontier) and returns a value that is then used to assess the relative efficiency of the portfolio under consideration. The GRS-W statistic is given as follows:

$$W = \left[\frac{\sqrt{1+\hat{\theta}_*^2}}{\sqrt{1+\hat{\theta}_p^2}}\right]^2 - 1 \equiv \psi^2 - 1 \qquad \text{(A-1)}$$

where $\hat{\theta}^*$ is the Sharpe measure of the ex post efficient portfolio (ratio of expected excess return to the standard deviation of the excess return) and $\hat{\theta}_p$ is the Sharpe measure of the test portfolio. Essentially θ is a slope measure ($\theta \equiv \overline{r}/\sigma$) with excess return ($\overline{r}$) and standard deviation of return (σ) and is the ray emanating from the origin on the Y-axis connecting to a portfolio in the first quadrant. Note that ψ cannot be less than 1 because $\hat{\theta}^*$ is the slope of the ex post frontier and is based on all the assets in the test (including portfolio *p*). To accept the efficiency of the test portfolio, ψ^2 should be close to 1. Larger values of ψ^2 imply portfolio inefficiency arising out of the increased distance between the test portfolio and the global MV efficient portfolio on the frontier ($W = \psi^2 - 1 \rightarrow 0$ implies efficiency). In other words, for values of W close to zero, the test portfolio cannot be called inefficient.

The test statistic is determined as follows:

$$[T(T-N-1)/N(T-2)] \times \left[\frac{\sqrt{1+\hat{\theta}_*^2}}{\sqrt{1+\hat{\theta}_p^2}}\right]^2 \equiv X_F \qquad \text{(A-2)}$$

It follows a F-distribution ~ $F(N, T-N-1)$, where *N* is the number of assets and *T* is the number of time series observations on the underlying asset returns.

H_0: Portfolio is efficient

The decision rule to reject H_0 is: Rej. H_0, iff. $F(X_F, N, T-N-1) <$ a threshold P-value

For the portfolios discussed in Exhibit 3, the various parameters required to determine the GRS-W statistic can be seen in the Exhibit A1.

The GRS test confirms that all three test portfolios are not efficient, relative to the tangency MAC portfolio. Please refer to Exhibit 4 for a graphical layout of these points. This essentially implies that these portfolios are not "close" to the efficient frontier. The works of Asness, Frazzini, and Pedersen [2012] and Clarke, de Silva and Thorley [2013] have discussed the "closeness" issue in their papers (in the context of risk-parity portfolios). Kale [2006] introduced the power-log functions to optimize portfolios with downside protection against the backdrop of prospect theory tenets.

Exhibit B1
Asset Weights for MAC Portfolios (%)

	Portfolio Beta (β)	Sharpe Ratio	EFA	EEM	SPY	TLT	IYR	GLD
MAC Portfolio (Asset Weights)	0.25	***0.736***	2.2	2.3	**30**	**30**	**5.6**	**30**
1/N MAC Port. (Asset Weights)	**0.68**	***0.471***	**16.7**	**16.7**	**16.7**	**16.7**	**16.7**	**16.7**
Russell 1000 Portfolio	1.00	***0.266***						

Appendix B

OPTIMAL PORTFOLIO WEIGHTS

Maillard, Roncalli, and Teiletche [2010] showed how ERC (equal-risk portfolios or risk-parity portfolios) are a good compromise between minimum-variance (which suffer from low volatility and high asset concentration) and $1/N$ portfolios, which do not account for individual asset volatilities or cross-correlations but do have equal asset weights.

Exhibit B1 shows the asset weights for the two MAC portfolios. These are based on the historical covariance matrix and do not need any estimates of ex ante return since we are operating in a min-variance framework. Also listed in the table are the portfolio betas and the respective Sharpe ratios. Please refer to Exhibit 4 to see the positioning of these points in the risk-return space. DeMiguel, Garlappi, and Uppal [2009] may have a point after all when they espouse the value of 1/N portfolio for which no ex ante estimates are necessary.

ENDNOTES

The author thanks the randomness of the capital markets for the formation of some of the ideas herein.

[1]The respective equivalent tickers are IWB, EFA, IWM, IWO, IWN, and DFBMX for the "all equity" portfolio.

[2]In this case, the minimum-variance process is used because we impose no prior on the vector of returns.

[3]This can be relaxed without loss of generality, but since most investors do not engage in short-selling or hold index funds, we apply the constraint. In any case, it would be an interesting extension of the study.

[4]The respective equivalent tickers are IWB, EFA, IWM, IWO, IWN, and DFBMX for the All Equity portfolio.

[5]Sharpe ratio calculated with $Rf = 0.7\%$ over the 2004–2013 period and estimated from BIL, the short-term U.S. Treasury. Given that the ETF BIL is traded, it has a annual volatility of 0.8% as well.

[6]Determined by the earliest starting date for this set of ETFs under consideration.

[7]As of May 22, 2013, the Dow was at an all time high of 15,307.

[8]Maillard, Roncalli, and Teiletche [2010] also provided the general case for $n > 2$ assets; see Qian [2006] for the $n = 2$ case.

[9]The flattening of the performance lines towards the top right-hand side of the graph in Exhibit 5 indicates the underperformance of bonds and gold relative to equities.

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